

12.3 | Stress, Strain, and Elastic Modulus

Learning Objectives

By the end of this section, you will be able to:

- Explain the concepts of stress and strain in describing elastic deformations of materials
- Describe the types of elastic deformation of objects and materials

A model of a rigid body is an idealized example of an object that does not deform under the actions of external forces. It is very useful when analyzing mechanical systems—and many physical objects are indeed rigid to a great extent. The extent to which an object can be *perceived* as rigid depends on the physical properties of the material from which it is made. For example, a ping-pong ball made of plastic is brittle, and a tennis ball made of rubber is elastic when acted upon by squashing forces. However, under other circumstances, both a ping-pong ball and a tennis ball may bounce well as rigid bodies. Similarly, someone who designs prosthetic limbs may be able to approximate the mechanics of human limbs by modeling them as rigid bodies; however, the actual combination of bones and tissues is an elastic medium.

For the remainder of this chapter, we move from consideration of forces that affect the motion of an object to those that affect an object's shape. A change in shape due to the application of a force is known as a deformation. Even very small forces are known to cause some deformation. Deformation is experienced by objects or physical media under the action of external forces—for example, this may be squashing, squeezing, ripping, twisting, shearing, or pulling the objects apart. In the language of physics, two terms describe the forces on objects undergoing deformation: *stress* and *strain*.

Stress is a quantity that describes the magnitude of forces that cause deformation. Stress is generally defined as *force per unit area*. When forces pull on an object and cause its elongation, like the stretching of an elastic band, we call such stress a **tensile stress**. When forces cause a compression of an object, we call it a **compressive stress**. When an object is being squeezed from all sides, like a submarine in the depths of an ocean, we call this kind of stress a **bulk stress** (or **volume stress**). In other situations, the acting forces may be neither tensile nor compressive, and still produce a noticeable deformation. For example, suppose you hold a book tightly between the palms of your hands, then with one hand you press-and-pull on the front cover away from you, while with the other hand you press-and-pull on the back cover toward you. In such a case, when deforming forces act tangentially to the object's surface, we call them 'shear' forces and the stress they cause is called **shear stress**.

The SI unit of stress is the pascal (Pa). When one newton of force presses on a unit surface area of one meter squared, the resulting stress is one pascal:

$$\text{one pascal} = 1.0 \text{ Pa} = \frac{1.0 \text{ N}}{1.0 \text{ m}^2}.$$

In the British system of units, the unit of stress is 'psi,' which stands for 'pound per square inch' (lb/in^2). Another unit that is often used for bulk stress is the atm (atmosphere). Conversion factors are

$$\begin{aligned} 1 \text{ psi} &= 6895 \text{ Pa} \quad \text{and} \quad 1 \text{ Pa} = 1.450 \times 10^{-4} \text{ psi} \\ 1 \text{ atm} &= 1.013 \times 10^5 \text{ Pa} = 14.7 \text{ psi}. \end{aligned}$$

An object or medium under stress becomes deformed. The quantity that describes this deformation is called **strain**. Strain is given as a fractional change in either length (under tensile stress) or volume (under bulk stress) or geometry (under shear stress). Therefore, strain is a dimensionless number. Strain under a tensile stress is called **tensile strain**, strain under bulk stress is called **bulk strain** (or **volume strain**), and that caused by shear stress is called **shear strain**.

The greater the stress, the greater the strain; however, the relation between strain and stress does not need to be linear. Only when stress is sufficiently low is the deformation it causes in direct proportion to the stress value. The proportionality constant in this relation is called the **elastic modulus**. In the linear limit of low stress values, the general relation between stress and strain is

$$\text{stress} = (\text{elastic modulus}) \times \text{strain}.$$

(12.33)

As we can see from dimensional analysis of this relation, the elastic modulus has the same physical unit as stress because strain is dimensionless.

We can also see from **Equation 12.33** that when an object is characterized by a large value of elastic modulus, the effect of stress is small. On the other hand, a small elastic modulus means that stress produces large strain and noticeable deformation. For example, a stress on a rubber band produces larger strain (deformation) than the same stress on a steel band of the same dimensions because the elastic modulus for rubber is two orders of magnitude smaller than the elastic modulus for steel.

The elastic modulus for tensile stress is called **Young's modulus**; that for the bulk stress is called the **bulk modulus**; and that for shear stress is called the **shear modulus**. Note that the relation between stress and strain is an *observed* relation, measured in the laboratory. Elastic moduli for various materials are measured under various physical conditions, such as varying temperature, and collected in engineering data tables for reference (**Table 12.1**). These tables are valuable references for industry and for anyone involved in engineering or construction. In the next section, we discuss strain-stress relations beyond the linear limit represented by **Equation 12.33**, in the full range of stress values up to a fracture point. In the remainder of this section, we study the linear limit expressed by **Equation 12.33**.

Material	Young's modulus $\times 10^{10} \text{ Pa}$	Bulk modulus $\times 10^{10} \text{ Pa}$	Shear modulus $\times 10^{10} \text{ Pa}$
Aluminum	7.0	7.5	2.5
Bone (tension)	1.6	0.8	8.0
Bone (compression)	0.9		
Brass	9.0	6.0	3.5
Brick	1.5		
Concrete	2.0		
Copper	11.0	14.0	4.4
Crown glass	6.0	5.0	2.5
Granite	4.5	4.5	2.0
Hair (human)	1.0		
Hardwood	1.5		1.0
Iron	21.0	16.0	7.7
Lead	1.6	4.1	0.6
Marble	6.0	7.0	2.0
Nickel	21.0	17.0	7.8
Polystyrene	3.0		
Silk	6.0		
Spider thread	3.0		
Steel	20.0	16.0	7.5
Acetone		0.07	
Ethanol		0.09	
Glycerin		0.45	
Mercury		2.5	
Water		0.22	

Table 12.1 Approximate Elastic Moduli for Selected Materials

Tensile or Compressive Stress, Strain, and Young's Modulus

Tension or compression occurs when two antiparallel forces of equal magnitude act on an object along only one of its dimensions, in such a way that the object does not move. One way to envision such a situation is illustrated in **Figure 12.18**. A rod segment is either stretched or squeezed by a pair of forces acting along its length and perpendicular to its cross-section. The net effect of such forces is that the rod changes its length from the original length L_0 that it had before the forces appeared, to a new length L that it has under the action of the forces. This change in length $\Delta L = L - L_0$ may be either elongation (when L is larger than the original length L_0) or contraction (when L is smaller than the original length L_0). Tensile stress and strain occur when the forces are stretching an object, causing its elongation, and the length change ΔL is positive. Compressive stress and strain occur when the forces are contracting an object, causing its shortening, and the length change ΔL is negative.

In either of these situations, we define stress as the ratio of the deforming force $F_{\perp}$ to the cross-sectional area A of the object being deformed. The symbol $F_{\perp}$ that we reserve for the deforming force means that this force acts perpendicularly to the cross-section of the object. Forces that act parallel to the cross-section do not change the length of an object. The definition of the tensile stress is

$$\text{tensile stress} = \frac{F_{\perp}}{A}. \quad (12.34)$$

Tensile strain is the measure of the deformation of an object under tensile stress and is defined as the fractional change of the object's length when the object experiences tensile stress

$$\text{tensile strain} = \frac{\Delta L}{L_0}. \quad (12.35)$$

Compressive stress and strain are defined by the same formulas, **Equation 12.34** and **Equation 12.35**, respectively. The only difference from the tensile situation is that for compressive stress and strain, we take absolute values of the right-hand sides in **Equation 12.34** and **Equation 12.35**.

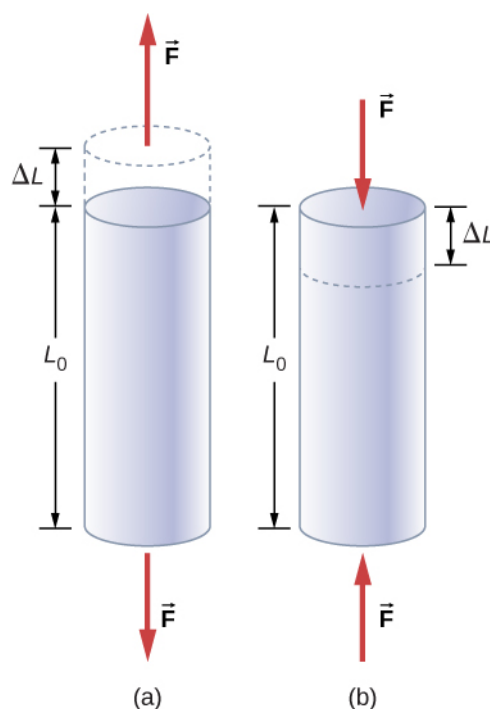


Figure 12.18 When an object is in either tension or compression, the net force on it is zero, but the object deforms by changing its original length L_0 . (a) Tension: The rod is elongated by ΔL . (b) Compression: The rod is contracted by ΔL . In both cases, the deforming force acts along the length of the rod and perpendicular to its cross-section. In the linear range of low stress, the cross-sectional area of the rod does not change.

Young's modulus Y is the elastic modulus when deformation is caused by either tensile or compressive stress, and is defined by **Equation 12.33**. Dividing this equation by tensile strain, we obtain the expression for Young's modulus:

$$Y = \frac{\text{tensile stress}}{\text{tensile strain}} = \frac{F_{\perp}/A}{\Delta L/L_0} = \frac{F_{\perp}}{A} \frac{L_0}{\Delta L}. \quad (12.36)$$

Example 12.7

Compressive Stress in a Pillar

A sculpture weighing 10,000 N rests on a horizontal surface at the top of a 6.0-m-tall vertical pillar **Figure 12.19**. The pillar's cross-sectional area is 0.20 m^2 and it is made of granite with a mass density of 2700 kg/m^3 . Find the compressive stress at the cross-section located 3.0 m below the top of the pillar and the value of the compressive strain of the top 3.0-m segment of the pillar.



Figure 12.19 Nelson's Column in Trafalgar Square, London, England. (credit: modification of work by Cristian Bortes)

Strategy

First we find the weight of the 3.0-m-long top section of the pillar. The normal force that acts on the cross-section located 3.0 m down from the top is the sum of the pillar's weight and the sculpture's weight. Once we have the normal force, we use **Equation 12.34** to find the stress. To find the compressive strain, we find the value of Young's modulus for granite in **Table 12.1** and invert **Equation 12.36**.

Solution

The volume of the pillar segment with height $h = 3.0$ m and cross-sectional area $A = 0.20$ m² is

$$V = Ah = (0.20 \text{ m}^2)(3.0 \text{ m}) = 0.60 \text{ m}^3.$$

With the density of granite $\rho = 2.7 \times 10^3$ kg/m³, the mass of the pillar segment is

$$m = \rho V = (2.7 \times 10^3 \text{ kg/m}^3)(0.60 \text{ m}^3) = 1.60 \times 10^3 \text{ kg}.$$

The weight of the pillar segment is

$$w_p = mg = (1.60 \times 10^3 \text{ kg})(9.80 \text{ m/s}^2) = 1.568 \times 10^4 \text{ N}.$$

The weight of the sculpture is $w_s = 1.0 \times 10^4$ N, so the normal force on the cross-sectional surface located 3.0 m below the sculpture is

$$F_{\perp} = w_p + w_s = (1.568 + 1.0) \times 10^4 \text{ N} = 2.568 \times 10^4 \text{ N}.$$

Therefore, the stress is

$$\text{stress} = \frac{F_{\perp}}{A} = \frac{2.568 \times 10^4 \text{ N}}{0.20 \text{ m}^2} = 1.284 \times 10^5 \text{ Pa} = 128.4 \text{ kPa}.$$

Young's modulus for granite is $Y = 4.5 \times 10^{10}$ Pa = 4.5×10^7 kPa. Therefore, the compressive strain at this position is

$$\text{strain} = \frac{\text{stress}}{Y} = \frac{128.4 \text{ kPa}}{4.5 \times 10^7 \text{ kPa}} = 2.85 \times 10^{-6}.$$

Significance

Notice that the normal force acting on the cross-sectional area of the pillar is not constant along its length, but varies from its smallest value at the top to its largest value at the bottom of the pillar. Thus, if the pillar has a uniform cross-sectional area along its length, the stress is largest at its base.



12.9 Check Your Understanding Find the compressive stress and strain at the base of Nelson's column.

Example 12.8

Stretching a Rod

A 2.0-m-long steel rod has a cross-sectional area of 0.30 cm^2 . The rod is a part of a vertical support that holds a heavy 550-kg platform that hangs attached to the rod's lower end. Ignoring the weight of the rod, what is the tensile stress in the rod and the elongation of the rod under the stress?

Strategy

First we compute the tensile stress in the rod under the weight of the platform in accordance with **Equation 12.34**. Then we invert **Equation 12.36** to find the rod's elongation, using $L_0 = 2.0 \text{ m}$. From **Table 12.1**,

Young's modulus for steel is $Y = 2.0 \times 10^{11} \text{ Pa}$.

Solution

Substituting numerical values into the equations gives us

$$\begin{aligned}\frac{F_{\perp}}{A} &= \frac{(550 \text{ kg})(9.8 \text{ m/s}^2)}{3.0 \times 10^{-5} \text{ m}^2} = 1.8 \times 10^8 \text{ Pa} \\ \Delta L &= \frac{F_{\perp}}{A} \frac{L_0}{Y} = (1.8 \times 10^8 \text{ Pa}) \frac{2.0 \text{ m}}{2.0 \times 10^{11} \text{ Pa}} = 1.8 \times 10^{-3} \text{ m} = 1.8 \text{ mm}.\end{aligned}$$

Significance

Similarly as in the example with the column, the tensile stress in this example is not uniform along the length of the rod. Unlike in the previous example, however, if the weight of the rod is taken into consideration, the stress in the rod is largest at the top and smallest at the bottom of the rod where the equipment is attached.



12.10 Check Your Understanding A 2.0-m-long wire stretches 1.0 mm when subjected to a load. What is the tensile strain in the wire?

Objects can often experience both compressive stress and tensile stress simultaneously **Figure 12.20**. One example is a long shelf loaded with heavy books that sags between the end supports under the weight of the books. The top surface of the shelf is in compressive stress and the bottom surface of the shelf is in tensile stress. Similarly, long and heavy beams sag under their own weight. In modern building construction, such bending strains can be almost eliminated with the use of I-beams **Figure 12.21**.

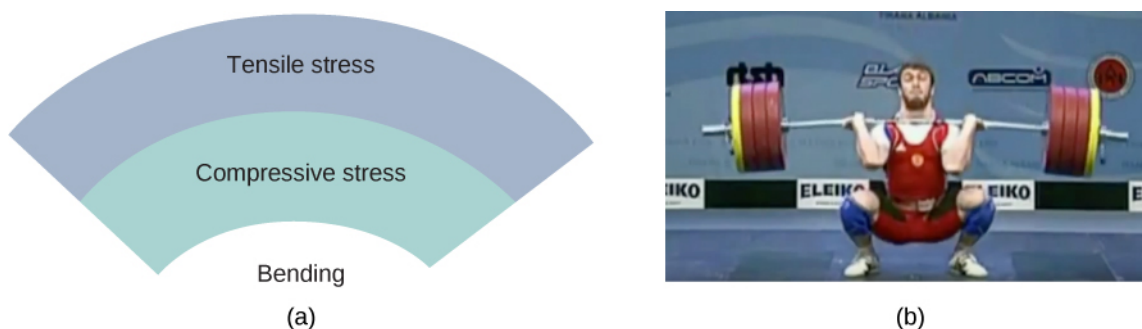


Figure 12.20 (a) An object bending downward experiences tensile stress (stretching) in the upper section and compressive stress (compressing) in the lower section. (b) Elite weightlifters often bend iron bars temporarily during lifting, as in the 2012 Olympics competition. (credit b: modification of work by Oleksandr Kocherzhenko)



Figure 12.21 Steel I-beams are used in construction to reduce bending strains. (credit: modification of work by “US Army Corps of Engineers Europe District”/Flickr)



A heavy box rests on a table supported by three columns. View this **demonstration** (<https://openstaxcollege.org//21movebox>) to move the box to see how the compression (or tension) in the columns is affected when the box changes its position.

Bulk Stress, Strain, and Modulus

When you dive into water, you feel a force pressing on every part of your body from all directions. What you are experiencing then is bulk stress, or in other words, **pressure**. Bulk stress always tends to decrease the volume enclosed by the surface of a submerged object. The forces of this “squeezing” are always perpendicular to the submerged surface **Figure 12.22**. The effect of these forces is to decrease the volume of the submerged object by an amount ΔV compared with the volume V_0 of the object in the absence of bulk stress. This kind of deformation is called bulk strain and is described by a change in volume relative to the original volume:

$$\text{bulk strain} = \frac{\Delta V}{V_0}. \quad (12.37)$$

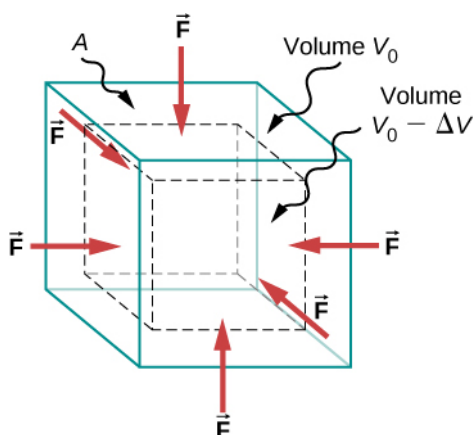


Figure 12.22 An object under increasing bulk stress always undergoes a decrease in its volume. Equal forces perpendicular to the surface act from all directions. The effect of these forces is to decrease the volume by the amount ΔV compared to the original volume, V_0 .

The bulk strain results from the bulk stress, which is a force $F_{\perp}$ normal to a surface that presses on the unit surface area A of a submerged object. This kind of physical quantity, or pressure p , is defined as

$$\text{pressure} = p \equiv \frac{F_{\perp}}{A}. \quad (12.38)$$

We will study pressure in fluids in greater detail in **Fluid Mechanics**. An important characteristic of pressure is that it is a scalar quantity and does not have any particular direction; that is, pressure acts equally in all possible directions. When you submerge your hand in water, you sense the same amount of pressure acting on the top surface of your hand as on the bottom surface, or on the side surface, or on the surface of the skin between your fingers. What you are perceiving in this case is an increase in pressure Δp over what you are used to feeling when your hand is not submerged in water. What you feel when your hand is not submerged in the water is the **normal pressure** p_0 of one atmosphere, which serves as a reference point. The bulk stress is this increase in pressure, or Δp , over the normal level, p_0 .

When the bulk stress increases, the bulk strain increases in response, in accordance with **Equation 12.33**. The proportionality constant in this relation is called the bulk modulus, B , or

$$B = \frac{\text{bulk stress}}{\text{bulk strain}} = -\frac{\Delta p}{\Delta V/V_0} = -\Delta p \frac{V_0}{\Delta V}. \quad (12.39)$$

The minus sign that appears in **Equation 12.39** is for consistency, to ensure that B is a positive quantity. Note that the minus sign (–) is necessary because an increase Δp in pressure (a positive quantity) always causes a decrease ΔV in volume, and decrease in volume is a negative quantity. The reciprocal of the bulk modulus is called **compressibility** k , or

$$k = \frac{1}{B} = -\frac{\Delta V/V_0}{\Delta p}. \quad (12.40)$$

The term ‘compressibility’ is used in relation to fluids (gases and liquids). Compressibility describes the change in the volume of a fluid per unit increase in pressure. Fluids characterized by a large compressibility are relatively easy to compress. For example, the compressibility of water is $4.64 \times 10^{-5}/\text{atm}$ and the compressibility of acetone is $1.45 \times 10^{-4}/\text{atm}$. This means that under a 1.0-atm increase in pressure, the relative decrease in volume is approximately three times as large for acetone as it is for water.

Example 12.9

Hydraulic Press

In a hydraulic press **Figure 12.23**, a 250-liter volume of oil is subjected to a 2300-psi pressure increase. If the compressibility of oil is $2.0 \times 10^{-5} / \text{atm}$, find the bulk strain and the absolute decrease in the volume of oil when the press is operating.

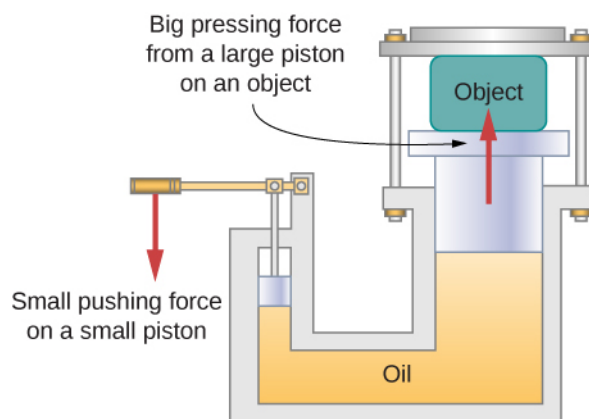


Figure 12.23 In a hydraulic press, when a small piston is displaced downward, the pressure in the oil is transmitted throughout the oil to the large piston, causing the large piston to move upward. A small force applied to a small piston causes a large pressing force, which the large piston exerts on an object that is either lifted or squeezed. The device acts as a mechanical lever.

Strategy

We must invert **Equation 12.40** to find the bulk strain. First, we convert the pressure increase from psi to atm, $\Delta p = 2300 \text{ psi} = 2300/14.7 \text{ atm} \approx 160 \text{ atm}$, and identify $V_0 = 250 \text{ L}$.

Solution

Substituting values into the equation, we have

$$\text{bulk strain} = \frac{\Delta V}{V_0} = \frac{\Delta p}{B} = k\Delta p = (2.0 \times 10^{-5} / \text{atm})(160 \text{ atm}) = 0.0032$$

$$\text{answer: } \Delta V = 0.0032 V_0 = 0.0032(250 \text{ L}) = 0.78 \text{ L.}$$

Significance

Notice that since the compressibility of water is 2.32 times larger than that of oil, if the working substance in the hydraulic press of this problem were changed to water, the bulk strain as well as the volume change would be 2.32 times larger.



12.11 Check Your Understanding If the normal force acting on each face of a cubical 1.0-m^3 piece of steel is changed by $1.0 \times 10^7 \text{ N}$, find the resulting change in the volume of the piece of steel.

Shear Stress, Strain, and Modulus

The concepts of shear stress and strain concern only solid objects or materials. Buildings and tectonic plates are examples of objects that may be subjected to shear stresses. In general, these concepts do not apply to fluids.

Shear deformation occurs when two antiparallel forces of equal magnitude are applied tangentially to opposite surfaces of a

solid object, causing no deformation in the transverse direction to the line of force, as in the typical example of shear stress illustrated in **Figure 12.24**. Shear deformation is characterized by a gradual shift Δx of layers in the direction tangent to the acting forces. This gradation in Δx occurs in the transverse direction along some distance L_0 . Shear strain is defined by the ratio of the largest displacement Δx to the transverse distance L_0

$$\text{shear strain} = \frac{\Delta x}{L_0}. \quad (12.41)$$

Shear strain is caused by shear stress. Shear stress is due to forces that act *parallel* to the surface. We use the symbol $F_{\parallel}$ for such forces. The magnitude $F_{\parallel}$ per surface area A where shearing force is applied is the measure of shear stress

$$\text{shear stress} = \frac{F_{\parallel}}{A}. \quad (12.42)$$

The shear modulus is the proportionality constant in **Equation 12.33** and is defined by the ratio of stress to strain. Shear modulus is commonly denoted by S :

$$S = \frac{\text{shear stress}}{\text{shear strain}} = \frac{F_{\parallel}/A}{\Delta x/L_0} = \frac{F_{\parallel}}{A} \frac{L_0}{\Delta x}. \quad (12.43)$$

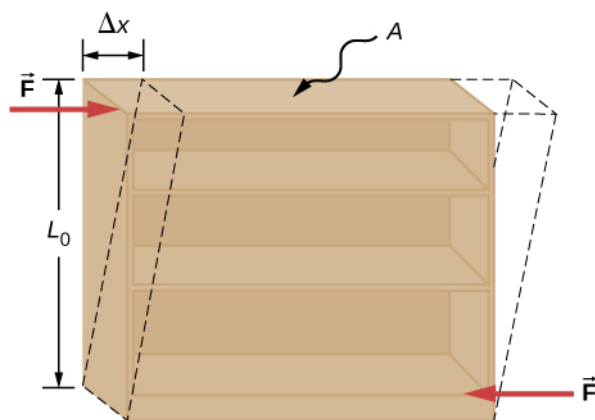


Figure 12.24 An object under shear stress: Two antiparallel forces of equal magnitude are applied tangentially to opposite parallel surfaces of the object. The dashed-line contour depicts the resulting deformation. There is no change in the direction transverse to the acting forces and the transverse length L_0 is unaffected. Shear deformation is characterized by a gradual shift Δx of layers in the direction tangent to the forces.

Example 12.10

An Old Bookshelf

A cleaning person tries to move a heavy, old bookcase on a carpeted floor by pushing tangentially on the surface of the very top shelf. However, the only noticeable effect of this effort is similar to that seen in **Figure 12.24**, and it disappears when the person stops pushing. The bookcase is 180.0 cm tall and 90.0 cm wide with four 30.0-cm-deep shelves, all partially loaded with books. The total weight of the bookcase and books is 600.0 N. If the person gives the top shelf a 50.0-N push that displaces the top shelf horizontally by 15.0 cm relative to the motionless bottom shelf, find the shear modulus of the bookcase.

Strategy

The only pieces of relevant information are the physical dimensions of the bookcase, the value of the tangential force, and the displacement this force causes. We identify $F_{\parallel} = 50.0 \text{ N}$, $\Delta x = 15.0 \text{ cm}$, $L_0 = 180.0 \text{ cm}$, and $A = (30.0 \text{ cm})(90.0 \text{ cm}) = 2700.0 \text{ cm}^2$, and we use **Equation 12.43** to compute the shear modulus.

Solution

Substituting numbers into the equations, we obtain for the shear modulus

$$S = \frac{F_{\parallel}}{A} \frac{L_0}{\Delta x} = \frac{50.0 \text{ N}}{2700.0 \text{ cm}^2} \frac{180.0 \text{ cm}}{15.0 \text{ cm}} = \frac{2}{9} \frac{\text{N}}{\text{cm}^2} = \frac{2}{9} \times 10^4 \frac{\text{N}}{\text{m}^2} = \frac{20}{9} \times 10^3 \text{ Pa} = 2.222 \text{ kPa}.$$

We can also find shear stress and strain, respectively:

$$\frac{F_{\parallel}}{A} = \frac{50.0 \text{ N}}{2700.0 \text{ cm}^2} = \frac{5}{27} \text{ kPa} = 185.2 \text{ Pa}$$

$$\frac{\Delta x}{L_0} = \frac{15.0 \text{ cm}}{180.0 \text{ cm}} = \frac{1}{12} = 0.083.$$

Significance

If the person in this example gave the shelf a healthy push, it might happen that the induced shear would collapse it to a pile of rubbish. Much the same shear mechanism is responsible for failures of earth-filled dams and levees; and, in general, for landslides.



12.12 Check Your Understanding Explain why the concepts of Young's modulus and shear modulus do not apply to fluids.

12.4 | Elasticity and Plasticity

Learning Objectives

By the end of this section, you will be able to:

- Explain the limit where a deformation of material is elastic
- Describe the range where materials show plastic behavior
- Analyze elasticity and plasticity on a stress-strain diagram

We referred to the proportionality constant between stress and strain as the elastic modulus. But why do we call it that? What does it mean for an object to be elastic and how do we describe its behavior?

Elasticity is the tendency of solid objects and materials to return to their original shape after the external forces (load) causing a deformation are removed. An object is **elastic** when it comes back to its original size and shape when the load is no longer present. Physical reasons for elastic behavior vary among materials and depend on the microscopic structure of the material. For example, the elasticity of polymers and rubbers is caused by stretching polymer chains under an applied force. In contrast, the elasticity of metals is caused by resizing and reshaping the crystalline cells of the lattices (which are the material structures of metals) under the action of externally applied forces.

The two parameters that determine the elasticity of a material are its *elastic modulus* and its *elastic limit*. A high elastic modulus is typical for materials that are hard to deform; in other words, materials that require a high load to achieve a significant strain. An example is a steel band. A low elastic modulus is typical for materials that are easily deformed under a load; for example, a rubber band. If the stress under a load becomes too high, then when the load is removed, the material no longer comes back to its original shape and size, but relaxes to a different shape and size: The material becomes permanently deformed. The **elastic limit** is the stress value beyond which the material no longer behaves elastically but becomes permanently deformed.

Our perception of an elastic material depends on both its elastic limit and its elastic modulus. For example, all rubbers are